Welcome!

Thanks for coming to this interactive workshop!

Please take a copy of handout 1 and introduce yourself to someone new. Then get your paper out and your pencils warmed up.

Ladies and gentlemen, start your counting… many fun problems await!

Combinatorics: The Breakfast of Mathletes

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Introductions

- Why are we here? To…
  - Do some (fun) math problems
  - Review some key combinatorial techniques
  - Discuss how combinatorics can help math teams
  - Discuss why students should learn combinatorics
- Who are you?
- Who am I?

Let's do some contest problems

- What do you notice?
  - How can you solve them?
  - What key techniques and methods emerge?
  - What connections do you see?
- What about the student perspective?
  - How could these help with other contest math?
  - How would these problems make students think?
  - What questions could they inspire students to ask?

So…

- What did you notice?
- What did you use?
- What did you learn?

Some elementary techniques

- Organized or strategic counting
- The multiplication principle
  - Leading to permutations and combinations
- Inclusion-exclusion
- Complementary counting
- Counting from multiple perspectives

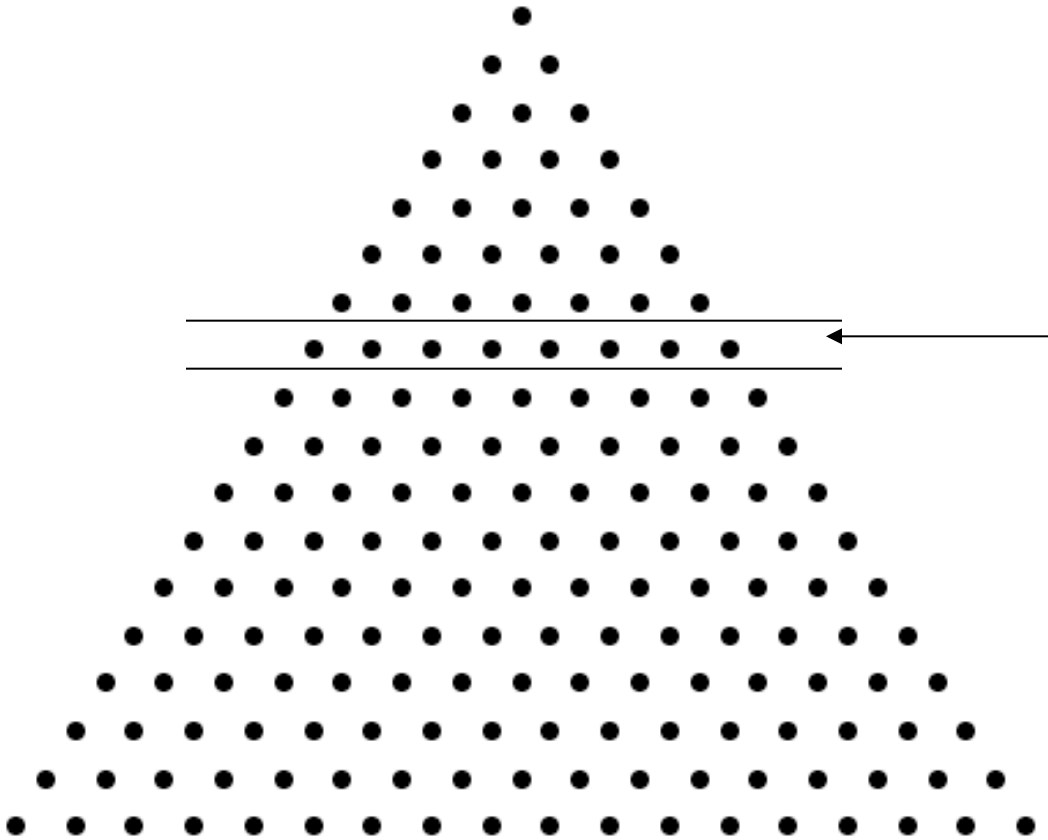
Why?

- Why combinatorics?
  - Wide range of complexity; concrete, and so accessible
  - Good ways to motivate key problem-solving strategies
  - Help engage, develop persistence, practice habits of mind
  - Support conjecture, generalization, mindful manipulation
- Why for mathletes?
  - Allow high-level problems without much “machinery”
  - Challenging problems often include combinatorics
  - Methods often neglected in the curriculum, but powerful
  - Contest problems can launch larger, deeper investigation

Some intermediate techniques

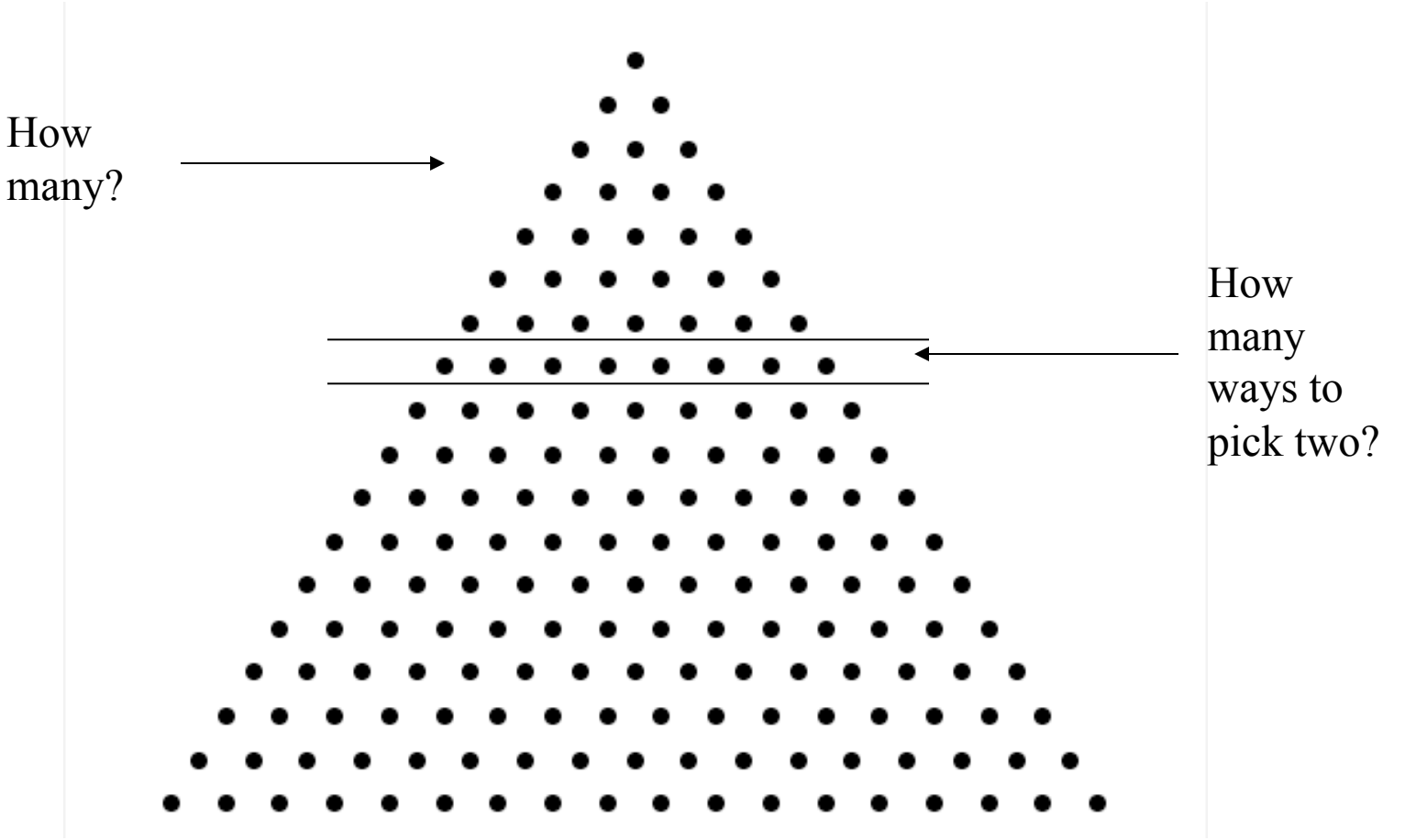
- Combining several elementary techniques
  - multiplication principle and combinations
  - multiplication principle with constraints
  - applying complementary counting
- Symmetries and intentional over-counting
- Working by cases
- Complex setups, probability, number theory

Don’t just count…

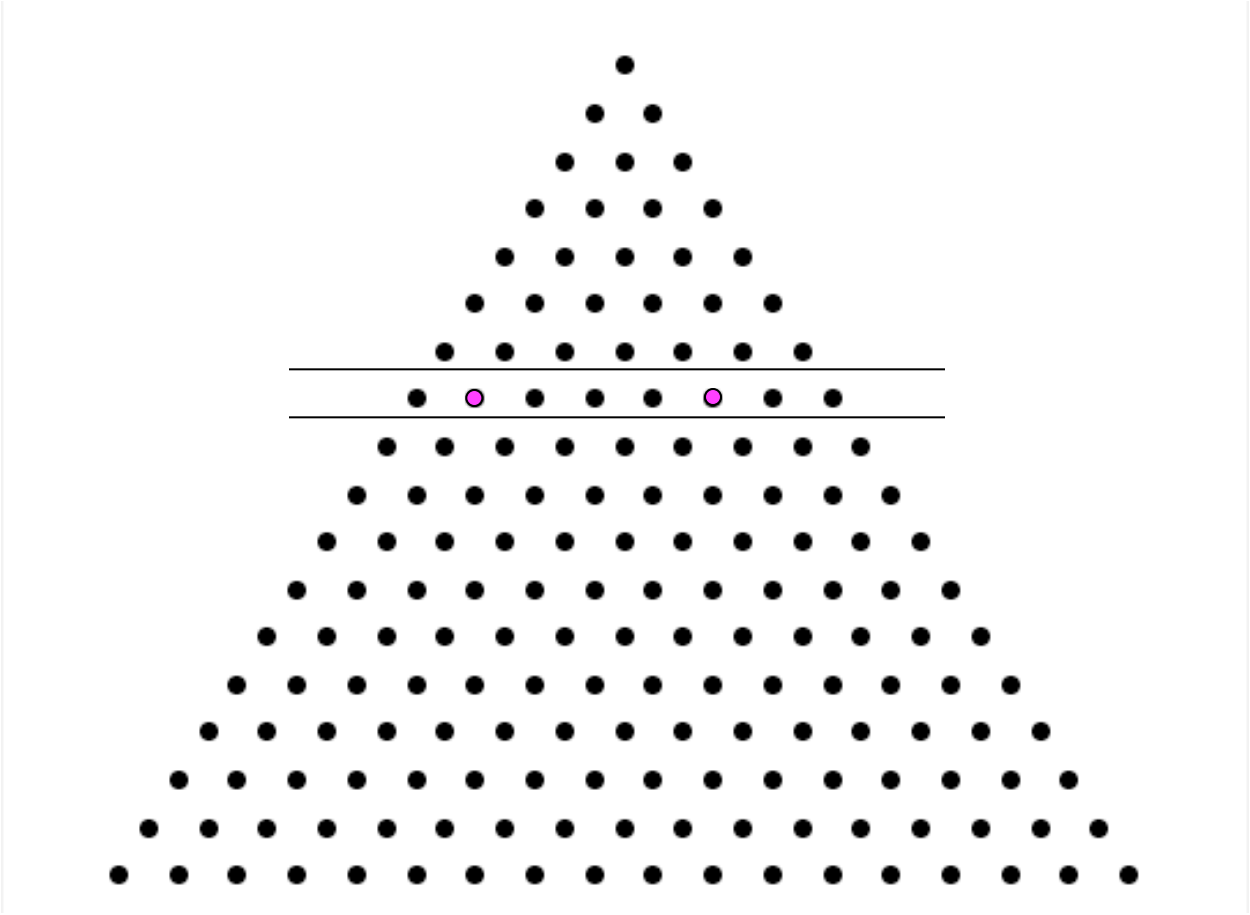


How many ways to pick two?

Don’t just count, count again



Proof without words



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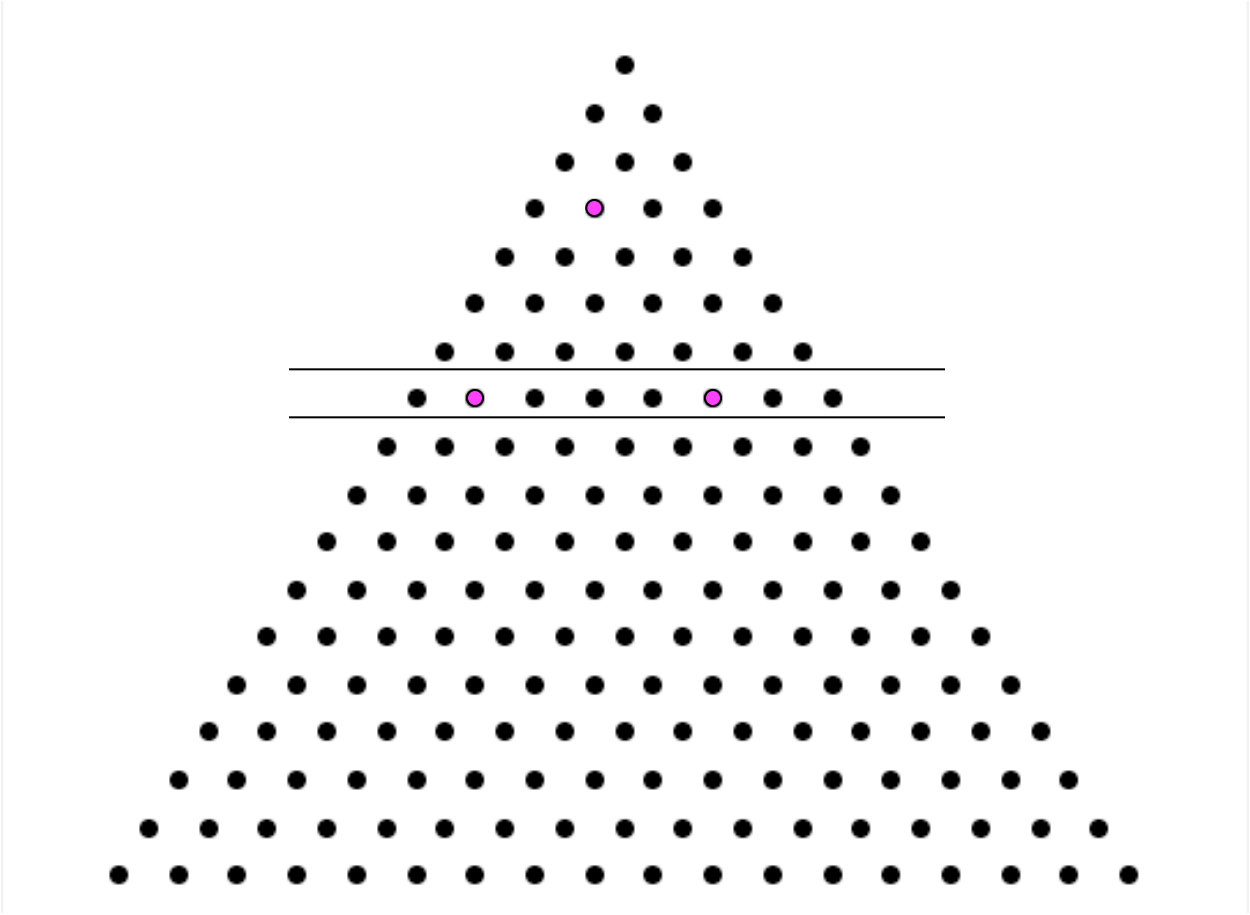


Why bother counting twice?

- How many ways are there to pick three people out of a group of six?
  - We routinely teach how to count once
  - Can we make it more general by counting again?
- Suppose there are five people.
  - How many ways are there to pick 0, 1, 2, 3, 4, 5?
  - What happens when Blaise walks in?

Combinatorial identities

- By generally counting one thing in two ways, you know those ways must be equivalent
- Corollary: Counting two things and putting them in 1-to-1 correspondence, means they are equivalent too
- This is a pretty powerful idea — and one much easier to introduce in a concrete context

Giving candies to kids

- What did you notice?
- What approaches did you use?
- Does it matter if you can tell the kids apart?
- (What if you can tell the candies apart?)
- Two different methods

Can candies lead us to an identity?

- Casework ("distinguished element") method:
  - Give Kid A 0 candies, have n for the other $k - 1$
  - Give Kid A 1 candy, have $n - 1$ left
  - …
  - Give Kid A n candies, have 0 left
- "Spacer" method: $n + k - 1$ choose $k - 1$ ways
- These are two fully general ways to count the same arrangements: must lead to an identity!

The hockey-stick identity

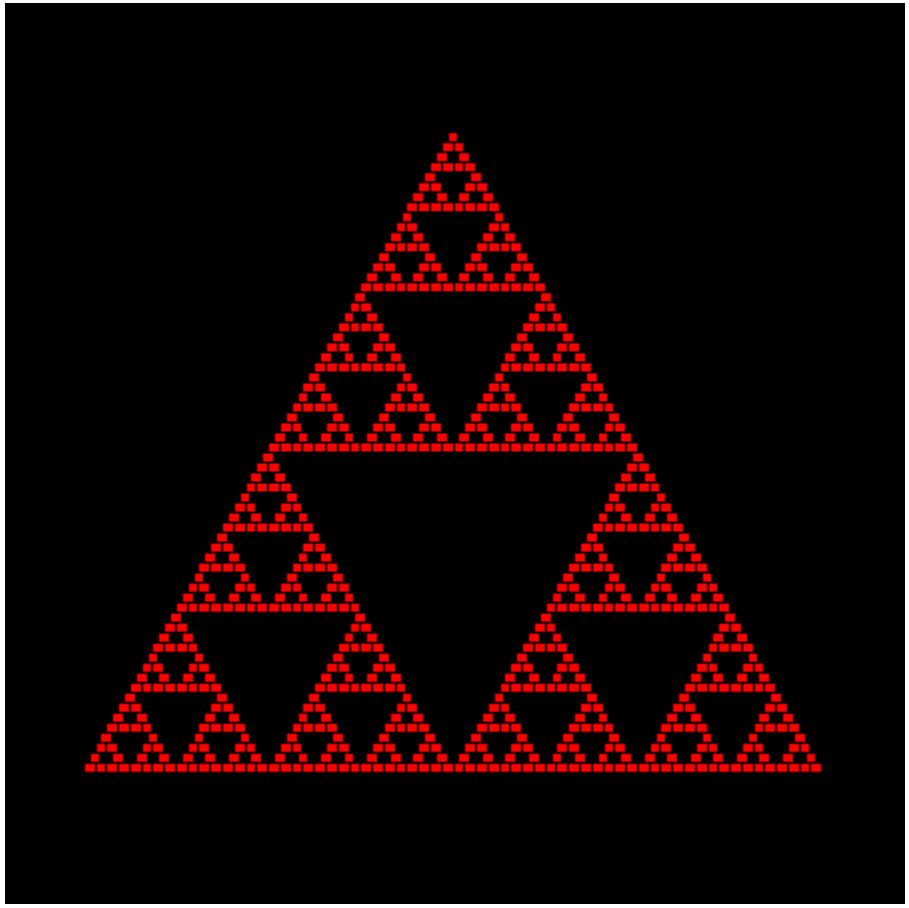
$$n, r \in \mathbb{N}, n > r, \sum_{i=r}^{n} \binom{i}{r} = \binom{n+1}{r+1}$$

Sum of terms from casework method

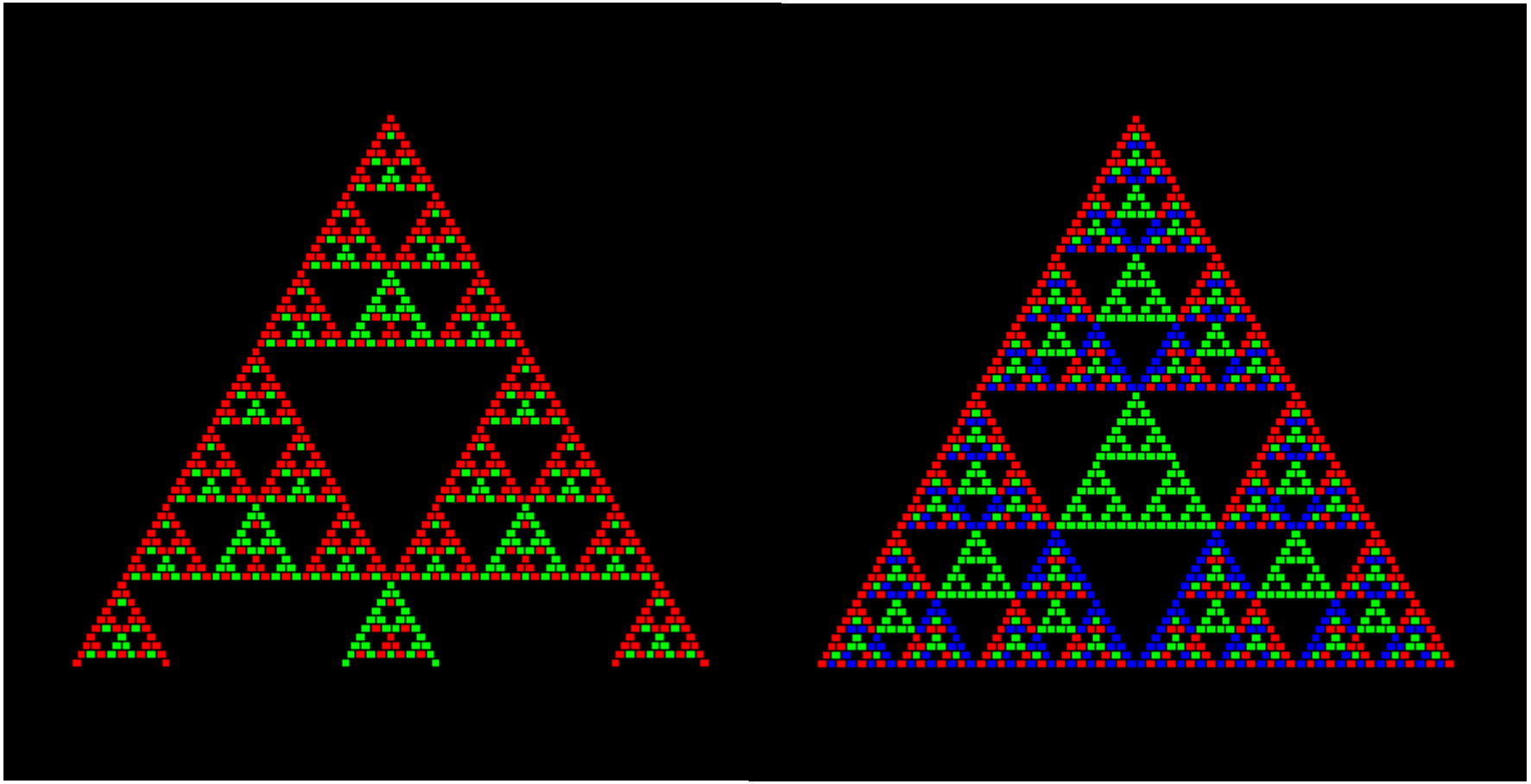
Spacer method

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
 & & 1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 & \\
 & 1 & & 8 & & 28 & & 56 & & 70 & & 56 & & 28 & & 8 & & 1 &
\end{array}$$

And about that triangle… even & odd



You can keep going: mod 3, mod 4



What are we noticing here?

- What's the numerical pattern?
- Where is the underlying pattern coming from?
- How could you…
  - Justify?
  - Generalize?
  - Extend?
  - Apply?
- A recursive relationship

This is all fun, but is it important?

- Why should you care about combinatorics?

- How do we want students to feel about math?
- How do we want them to think about math?

- Also, is combinatorics good for something?

How do we want students to feel?

- “The mind is not a vessel to be filled, but a fire to be ignited.” – *Plutarch*

- “If you want to build a ship, don't drum up people together to collect wood and don't assign them tasks and work, but rather teach them to long for the endless immensity of the sea.” – *Antoine de Saint-Exupery*

How Mathematicians Think

"If we wish to talk about mathematics in a way that includes acts of creativity and understanding, then we must be prepared to adopt a different point of view from the one in most books about mathematics and science. **When mathematics is viewed as content, it is lifeless and static**…"

– William Byers

Imagine Math Day at Harvey Mudd

"[We need] opportunities to remind [our]selves why teaching, learning, and creating math can be useful, rewarding and fulfilling. [We] need to be aware of the **powerful role that math can play in the lives of [our] students**… because [math can] be an effective vehicle for teaching students valuable 'habits of mind.'"

– Yong and Orrison, *MAA Focus*, 2008

Problem-solving habits of mind

- Pólya's process (*How to Solve It*):
  - Understand, plan, solve, check
  - Looking for patterns and connections
  - Developing heuristics
  - "work backwards", "try a simpler case", etc.
- Developing flexible thinkers
- Justification emerges naturally

Developing problem-solving skills

- A few principles, many *connected* techniques
- Students learn that *experience* solving really contributes to their skill (growth mindset)
- Helps orderly, algebraic thinking, and can address and motivate algebraic fluency too
- Develops inductive thinking (conjecturing) as well as deductive thinking (proof), and these problems often connect them really well

What cognitive habits do we seek?

- Questioning
- Forming conjectures
- Trying a simpler problem
- Seeing similarities among related problems
- Finding connections
- Generalizing

Questions and habits to develop

- Is there another way to look at this situation? Is there a different question to ask?
- How could I make this problem…
  - Simpler or easier?
  - More general, extended, or harder?
- How does this content, or this method, connect to other problems or other topics?

How does this open up our classes?

- Concrete, and so, accessible
  - Low threshold, high (or no) ceiling
- Naturally permits multiple solution methods
- Connected to multiple key habits of mind
- Naturally problem-centered, student-centered
- More students can succeed at math if there are more ways to be successful (Cohen, Silver)

How will this affect your teaching?

- What's one idea you've gained
  or one connection you've made?
- What's one thing you're going to try?
- What's one thing you'll tell someone about?

Thank you!

- Please email with feedback, questions, ideas, comments, and more problems and resources!
- I'm happy to send you these slides and our handouts (and more, from a longer version)

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